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Generation of instability waves in flows separating from smooth surfaces

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This paper analyses the coupling between an imposed disturbance and an instability wave that propagates downstream on a shear layer which emanates from a separation point on a smooth surface. Since the wavelengths of the most-amplified instability waves will generally be small compared with the streamwise body dimensions, the analysis is restricted to this 'high-frequency' limit and the solution is obtained by using matched asymptotic expansions. An 'inner' solution, valid near the separation point, is matched onto an outer solution, which represents an instability wave on a slowly diverging mean flow. The analysis relates the amplitude of this instability to that of the imposed disturbance.

1. Introduction

It has been known for some time that acoustic excitation can have a strong effect on separated flows over airfoils at high angles of attack – completely eliminating the separated regions in some cases. There are some instances where this can be attributed to an upstream boundary-layer transition promoted by the acoustic excitation (e.g. the experiments of Mueller & Batill 1982), but recent experiments of Ahuja, Whipkey & Jones (1983) show that periodic acoustic excitation can greatly reduce the size of the separated region even when the unexcited boundary layer is already turbulent. (Glass beads were used to trip the unexcited boundary layer in this experiment.)

The magnitude of the effect can be seen from Ahuja, Whipkey & Jones' photographs reproduced here as figure 1. Their visualization studies also revealed the presence of strong large-scale coherent motions on the separated regions of the excited flows. It is therefore possible that the diminished separation resulted from enhanced entrainment promoted by instability waves that were triggered on the separated shear layers by the acoustic excitation.

In this paper we analyse the coupling between an external disturbance and an instability wave on a shear layer that emanates from a smooth surface (shown schematically in figure 2), or, in the words of Morkovin (1969), determine the *receptivity* of this flow to the external 'forcing'. Recall that an instability wave, being an eigensolution, is usually indeterminate to within a multiplicative constant. The determination of this constant, which, following Tam (1971), we refer to as the *coupling coefficient*, is the central purpose of this paper.

We suppose that the disturbance is of small amplitude and has harmonic time dependence and that the mean-flow Reynolds number is large. Even though we restrict the analysis to two-dimensional incompressible motion, the relevant separated flow is still quite complex. However, we treat the unsteady motion as a linear perturbation about an appropriate steady flow and recall that there is now considerable evidence to support the contention (Sychev 1972; Messiter & Enlow 1973;

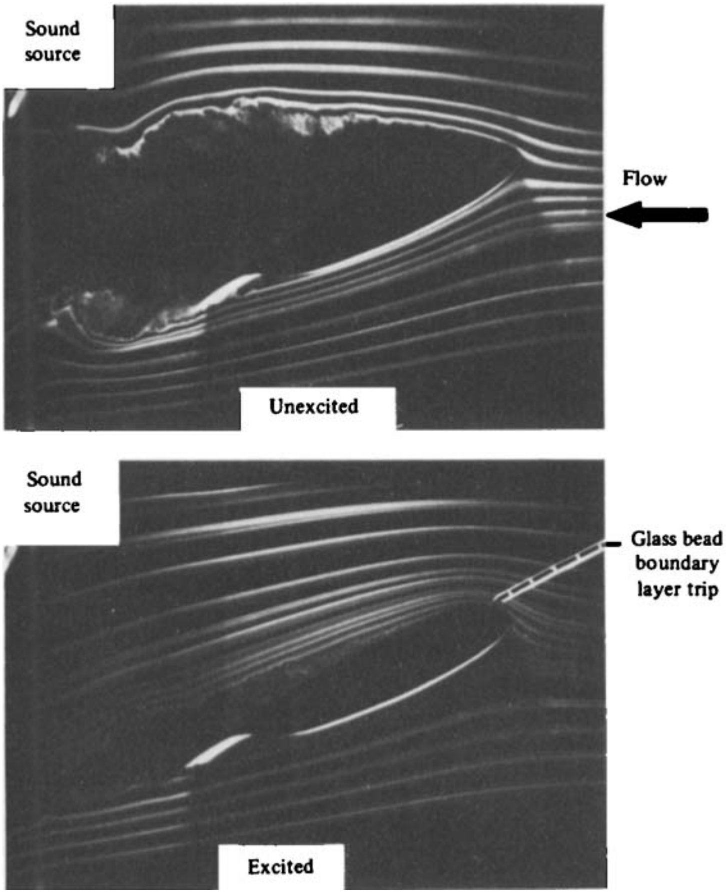

FIGURE 1. Acoustic control of flow separation as visualized in a smoke tunnel by Ahuja *et al.* (1983). Free-stream velocity = 13 m/s, angle of attack = 16°, excitation frequency = 640 Hz.

Smith, 1977) that the steady solution for grossly separated high-Reynolds-number flows is given, to leading order in Reynolds number, by the Kirchoff (1869) free-streamline solution, as was proposed many years ago by Imai (1953), while recent work of Smith & Daniels (1981) demonstrates the validity of free-streamline theory even for smaller-scale separations that involve recirculation eddies.

We, therefore, use the Kirchoff solution to describe the basic steady flow. It involves a constant pressure (and consequently zero velocity) wake separated from the region of flow by a dividing streamline across which the tangential velocity changes discontinuously. The Kirchoff solution is not, however, unique, since the location of its separation point can be chosen arbitrarily. There is one location of this point, referred to as a *Brillouin point*, where the curvatures of the solid surface and separated streamline are equal to one another at the separation point.

The actual location of the laminar flow separation point can be determined by treating the Kirchoff solution as the lowest-order approximation in a systematic asymptotic expansion in inverse powers of the Reynolds number and carrying the analysis to second order. Consideration of viscous effects in the vicinity of the separation point (Sychev 1972; Messiter & Enlow 1973; Smith 1977) shows that the flow has the well-known 'triple-deck' structure of Stewartson (1969) and Messiter

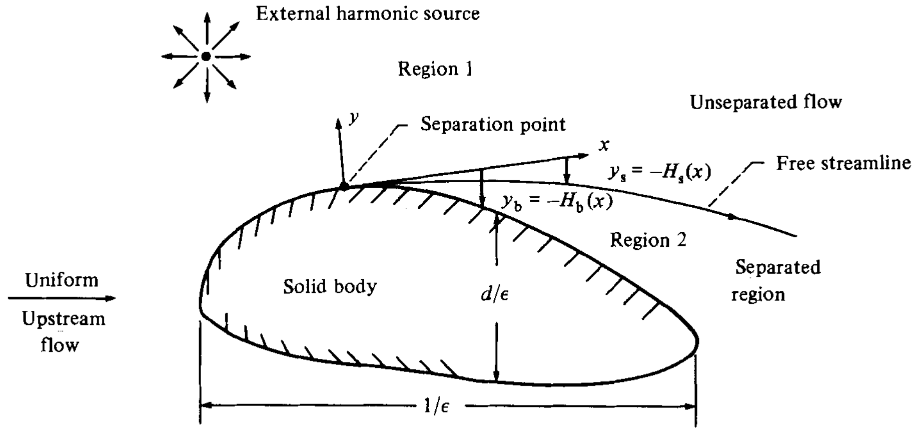

FIGURE 2. Configuration of separated flow.

(1970) in this region. These studies also show that the flow separates near its Brillouin point – the distance between the separation and Brillouin points being of the order of the inverse Reynolds number to the $\frac{1}{16}$ power (Smith 1977). Moreover, a recent study of Sychev & Sychev (1980) suggests that the Kirchoff free-streamline solution also applies to turbulent separations, but that the separation point then remains at a finite distance downstream of its Brillouin point as the Reynolds number becomes infinite (separation upstream of the Brillouin point is geometrically impossible because the dividing streamline would then cut the body surface; Imai 1953). Finally, Cheng & Smith (1982) show that even a laminar flow separation point will remain at a finite distance from the Brillouin point as the Reynolds-number becomes infinite when the body is sufficiently thin (i.e. when its thickness is of the order of the Reynolds number to the minus $\frac{1}{16}$ power).

Crighton & Leppington (1974), Rienstra (1981) and Orszag & Crow (1970) studied the excitation of instability waves on shear layers emanating from sharp trailing edges. They analysed the small-amplitude harmonic motion imposed on a steady flow over a semi-infinite flat plate with zero mean velocity on one side and uniform mean velocity on the other; so that a velocity-discontinuity shear layer extended downstream from the trailing edge. Their calculations show that this problem has a solution that (1) remains bounded at large distances from the trailing edge and (2) has a square-root singularity at that edge. But they also show that the problem possesses an eigensolution associated with the spatially growing instability wave on the downstream velocity discontinuity shear layer and that this eigensolution has a corresponding square-root singularity at the edge. An arbitrary constant multiple of the latter can therefore be added to the solution that is bounded at infinity, and the constant can then be adjusted to eliminate the singularity between the two solutions, i.e. to satisfy a 'Kutta' condition at the edge. Calculations involving viscosity (Daniels 1977) confirm the validity of the Kutta condition for sufficiently high-Reynolds-number laminar flows, and carefully controlled experiments (Bechert & Pfizenmaier 1975) indicate that the Kutta condition is also satisfied in many real flows.

The present study reveals that flows separating from smooth surfaces exhibit similar behaviour when their separation points are not too close to their Brillouin points, i.e. there is a forced inviscid solution that does not involve an instability wave and possesses a square-root singularity at the separation point and an inviscid eigensolution that *does* involve a downstream-propagating instability wave and possesses a corresponding singularity at the separation point. One might therefore conjecture that the correct inviscid solution to the present problem is also given by the linear combination of these two solutions that is non-singular at the separation point, i.e. that satisfies a 'Kutta' condition there. This requirement completely determines the amplitude of the instability wave in terms of the 'forcing' amplitude, i.e. determines the *coupling coefficient.*

This coefficient is then given by a very simple formula ((6.2) below), which is independent of the precise nature of the forcing. We regard it as a central result of this work. But, since the Kutta condition has not even been previously proposed (not to mention verified) for the present situation, it is important to demonstrate that it applies in at least one special case.

We do this for the case of 'quasi-steady' laminar separation by analysing the viscous (and nonlinear) effects in the vicinity of the separation point. By 'quasi-steady' we mean that the time enters only as a parameter, in the triple-deck region which, as we shall establish subsequently, surrounds the separation point in this case. This will be the case whenever the Strouhal number is much smaller than the Reynolds number to the $\frac{1}{4}$ power, which is the situation most likely to be encountered in practice. Our analysis reveals that it is the viscous (and nonparallel-flow) effects resulting from the steep velocity gradients in the vicinity of the separation point that physically produce the coupling between the instability wave and the external forcing. The results strongly suggest that the Kutta condition will *not* be satisfied when the Strouhal number is of the order of the $\frac{1}{4}$ power of the Reynolds number. This is in marked contrast with the sharp-training-edge case, where the Kutta condition is still satisfied at these larger Strouhal numbers (Daniels 1977; Brown and Daniels 1975).

Since the present analysis is fairly complex – though the mathematics are relatively straightforward – I will outline the steps in some detail. The general problem is formulated in §2. Since it turns out that the characteristic wavelength of the most rapidly growing instability wave is small compared with the streamwise body dimensions, I have restricted the analysis to this high-frequency limit. Notice that we do not require the acoustic wavelengths (which is in fact infinite since the flow is incompressible) to be small. In §3 we develop the scaling and appropriate form of the relevant asymptotic expansion for the region of flow where the wavelength of the instability wave is of the same order as the transverse dimension of the separated region. Here it is necessary to consider separately (1) the case where the transverse body dimension is large compared with the characteristic wavelength of the unsteady motion, and (2) the 'slender-body' case where the wavelength is of the same order as that dimension. In case (1) we have to distinguish between the two subcases where separation does and does not occur at a Brillouin point – though, as we shall show subsequently, the former subcase is unlikely to be realized in any actual flow.

The instability wave solution is constructed in §4. The scaling developed in §3 is used in §4.1 to construct a solution that describes the unsteady motion in the region where the characteristic wavelength of that motion is of the order of the transverse dimension of the separated region. This solution behaves like a Kelvin–Helmholtz instability wave on a vortex sheet that is nearly parallel to an adjacent wall. But, since this wave propagates over relatively large streamwise distances, the effect of the slowly varying mean flow must be accounted for even at the lowest order of approximation. This is done by using the method of multiple scales (Nayfeh 1973). The resulting solution is given by a very simple closed-form expression. It is re-expanded (in §4.2) for positions close to the separation point and is found to be invalid there, i.e. it does not provide a valid solution to the linearized inviscid equations in a sufficiently small semicircular region about the separation point. Since this is precisely the region where the coupling between the forcing and the instability wave occurs, we construct a new 'inner' solution for this region in §4.3. At this point we restrict the analysis to the case where the separation does not occur at a Brillouin point.

The geometry is locally flat in the inner region, and finding the inner solution amounts to solving a boundary-value problem for an analytic function w (namely the complex-conjugate velocity) of the complex coordinate variable z in the upper-half z-plane subject to certain boundary conditions specified along the real axis. Thus the imaginary part of w is required to vanish on the negative real axis (corresponding to the zero-normal-velocity wall boundary condition), and the imaginary part of w is given as a linear combination of derivatives of its real part on the positive real axis.

The problem is solved by analytically continuing the positive-real-axis boundary condition into the upper-half z-plane to obtain a third-order ordinary differential equation with independent variable z and dependent variable w. This equation is solved in closed form (even though it has variable coefficients) and it is shown that one of its three solutions automatically satisfies the upstream boundary condition on the negative real axis.

In §4.4 we show that this solution also matches, in the matched-asymptotic-expansion sense, onto the outer instability-wave solution constructed in §4.1 and is therefore the correct 'inner' solution. The corresponding composite expansion then provides a complete (uniformly valid) solution for the instability wave. This result, which does not involve external forcing, is also an eigensolution to the problem. It turns out that it has a square-root singularity at the separation point.

We then construct (in §5) a particular solution to the problem which does involve external forcing but does not involve an instability wave and therefore remains bounded far downstream in the flow. We assume that it is produced by an external source located within several wavelengths of the separation point. An explicit formula is obtained in §5.1 by using relatively straightforward complex-variable methods. It is re-expanded (in §5.1) for points close to the separation point and shown to be a uniformly valid solution to the linearized inviscid equations in that region. It turns out that this solution also possesses a square-root singularity at the separation point.

The coupling between the forced solution and the instability wave is discussed in §6. We first consider the consequences of imposing a Kutta condition to eliminate the singularity in the linearized inviscid solution. This determines the 'coupling coefficient' as already indicated.

Even though our linearized solution is a uniformly valid approximation to the linearized equations, it is not necessarily a uniformly valid solution to the full nonlinear equations. In fact, we show that the singularity of the linearized solution can be eliminated by accounting for the nonlinear effects that result from the motion of the separation point. The Kutta-condition solution emerges as a special case corresponding to negligibly small motion of that point. The relation between that motion and the external forcing is, in the general case, determined by the viscous effects in the vicinity of the separation point.

The validity of the analysis is assessed in §7. It is shown that the non-Brillouin point separation considered herein even applies to laminar boundary layers on blunt bodies unless the Reynolds number is extremely large – *larger, in fact, than any*

Reynolds number at which one could reasonably expect the upstream boundary layer to remain laminar!

Finally, we consider (in § 8) the viscous effects near the separation point and obtain an explicit expression for the separation-point motion (in terms of the forcing) for the special case of quasi-steady laminar separation alluded to above. It turns out that the motion is indeed small and the Kutta condition is therefore satisfied in this important case.

2. Formulation

We consider a two-dimensional incompressible flow over a two-dimensional body of characteristic streamwise dimension l. The upstream flow is assumed to be uniform and steady with velocity U_s and the Reynolds number $Re = U_s l/\nu$, where ν is the kinematic viscosity, is assumed to be large. We allow the flow to be separated, but require that all of its unsteadiness be the direct result of an imposed harmonic motion whose frequency we denote by ω and whose velocity we require to be everywhere small compared with both U_s and ωl. We suppose that the time has been non-dimensionalized by ω^{-1} and that all velocities, pressures and lengths are non-dimensionalized by U_s, ρU_s^2 and U_s/ω respectively.

We denote the pressure and velocity of the mean flow by P and $\boldsymbol{U} = \{U, V\}$ respectively and, as indicated in §1, represent it by the appropriate Kirchoff free-streamline solution. Then the complex-conjugate velocity $\zeta = U - \mathrm{j}V$, where $\mathrm{j} = \sqrt{-1}$,† is an analytic function of $z = x + \mathrm{j}y$ in the unseparated region (region 1 in figure 2), is equal to zero within the separated region (region 2 in figure 2) and has magnitude $|\zeta_0| \equiv (U^2 + V^2)^{\frac{1}{2}} = 1$ on the free streamline s that separates these two regions.

The unsteady motion is also assumed to be inviscid. Then it can be treated as a potential flow in both regions 1 and 2 and, since it can be linearized about the mean flow in the former region, will have harmonic time dependence with frequency ω. The pressure fluctuation $\alpha_0 p_1(\boldsymbol{x}, t)$ and the fluctuating part $\alpha_0 \boldsymbol{u}_1(\boldsymbol{x}, t)$ of the velocity $\boldsymbol{U}(\boldsymbol{x}) + \alpha_0 \boldsymbol{u}_1(\boldsymbol{x}, t)$ in region 1 can therefore be written as

$$p_1 = -\frac{\mathrm{D}}{\mathrm{D}t}[\phi_1(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}t}] = [(\mathrm{i} - \boldsymbol{U}\cdot\boldsymbol{\nabla})\,\phi_1]\,\mathrm{e}^{-\mathrm{i}t} \tag{2.1}$$

and

$$\boldsymbol{u}_1 = \{u_1, v_1\} = \boldsymbol{\nabla}[\phi_1(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}t}] \tag{2.2}$$

respectively, where α_0 denotes the small constant amplitude of the fluctuation $\boldsymbol{x} = \{x, y\}$,

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}$$

is the convective derivative based on the mean-flow velocity, and

$$\nabla^2 \phi_1 = 0. \tag{2.3}$$

Similarly, the unsteady pressure $\alpha_0 p_2$ and velocity $\alpha_0 \boldsymbol{u}_2$ in region 2 can be written as

$$p_2 = \mathrm{i}\phi_2(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}t} \tag{2.4}$$

and

$$\boldsymbol{u}_2 = \{u_2, v_2\} = \boldsymbol{\nabla}[\phi_2(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}t}] \tag{2.5}$$

respectively, where

$$\nabla^2 \phi_2 = 0. \tag{2.6}$$

† Note that we use both i and j to denote $\sqrt{-1}$, but it is necessary to keep the complex variables (involving i) associated with the harmonic time dependence separate from those (involving j) associated with the spatial dependence of the complex variable z.

The position of the body surface can be described by an equation of the form

$$y - H_b(x) = 0, \tag{2.7}$$

and the position of the free streamline by an equation of the form

$$y - H_s(x) - \alpha_0 h_s(x, t) = 0, \tag{2.8}$$

where $\alpha_0 h_s$ is small compared with the mean position H_s. Then, since the free surface is a fluid material surface for both the separated and unseparated flow, it follows that

$$\left.\begin{aligned} &\left(\frac{\mathrm{D}}{\mathrm{D}t} + \alpha_0 \boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\right)(y - H_s - \alpha_0 h_s) = 0, \\ &\left(\frac{\partial}{\partial t} + \alpha_0 \boldsymbol{u}_2 \cdot \boldsymbol{\nabla}\right)(y - H_s - \alpha_0 h_s) = 0 \end{aligned}\right\} \quad \text{at} \quad y = H_s + \alpha_0 h_s, \tag{2.9}$$

and that the pressure is continuous across s.

Transferring these to the mean position $y = H_s$ of the free surface by expanding U, V, and P in a Taylor series about this position and then linearizing the results about the mean-flow pressure-continuity condition and the mean-flow tangency condition

$$V = \frac{\mathrm{d}H_s}{\mathrm{d}x} U \quad \text{at} \quad y = H_s, \tag{2.10}$$

we obtain

$$\frac{\mathrm{D}h_s}{\mathrm{D}t} - h_s\left(\frac{\partial V}{\partial y} - \frac{\mathrm{d}H_s}{\mathrm{d}x}\frac{\partial U}{\partial y}\right) = v_1 - \frac{\mathrm{d}H_s}{\mathrm{d}x} u_1 \quad \text{at} \quad y = H_s, \tag{2.11}$$

$$\frac{\partial h_s}{\partial t} = v_2 - \frac{\mathrm{d}H_s}{\mathrm{d}x} u_2 \quad \text{at} \quad y = H_s, \tag{2.12}$$

$$p_1 + h_s \frac{\partial P}{\partial y} = p_2 \quad \text{at} \quad y = H_s. \tag{2.13}$$

Using (2.10), Bernoulli's equation $2P + V^2 + U^2 = U_s^2$ and the fact that U and V satisfy the Cauchy–Riemann equations

$$\frac{\partial U}{\partial x} = -\frac{\partial V}{\partial y} \tag{2.14}$$

and

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}, \tag{2.15}$$

we obtain upon eliminating h_s between (2.11)–(2.13)

$$\left\{\frac{\mathrm{D}}{\mathrm{D}t} + \frac{1}{2U}\left[\frac{\partial}{\partial x}(U^2 + V^2)\right]\right\}\left(v_2 - \frac{\mathrm{d}H_s}{\mathrm{d}x} u_2\right) = \frac{\partial v_1}{\partial t} - \frac{\mathrm{d}H_s}{\mathrm{d}x}\frac{\partial u_1}{\partial t}, \tag{2.16}$$

$$\frac{\partial p_2}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial y}(U^2 + V^2)\right]\left(v_2 - \frac{\mathrm{d}H_s}{\mathrm{d}x} u_2\right) = \frac{\partial p_1}{\partial t} \tag{2.17}$$

at $y = H_s$.

The boundary condition on the solid surface is

$$v_{1,2} = \frac{\mathrm{d}H_b}{\mathrm{d}x} u_{1,2} \quad \text{on} \quad y = H_b. \tag{2.18}$$

3. High-frequency expansion

We have already indicated that our interest is in the high-frequency limit where the characteristic wavelength U_s/ω of the unsteady motion is small compared with the streamwise dimension l of the body, i.e. where

$$\epsilon \equiv U_s/\omega l \ll 1. \tag{3.1}$$

Since the mean flow varies on the scale l of the body, it will depend on x and y only through the slow variable

$$\bar{z} \equiv \bar{x}+\mathrm{j}\bar{y} \equiv \epsilon x+\mathrm{j}\epsilon y, \tag{3.2}$$

i.e. the mean velocity will be given by

$$\zeta_0 \equiv U-\mathrm{j}V = \bar{F}(\bar{z}) \tag{3.3}$$

independently of ϵ, and the mean free-streamline position and body-surface location will be given by

$$y = H_s = \frac{1}{\epsilon}\bar{H}_s(\bar{x}), \tag{3.4}$$

$$y = H_b = \frac{1}{\epsilon}\bar{H}_b(\bar{x}) \tag{3.5}$$

respectively, where $\bar{H}_s$ and $\bar{H}_b$ are, in general, $O(1)$ quantities that do not depend on ϵ.

Near the separation point, $\bar{z} = 0$, the thickness of the separated region will be small compared with the spatial scale U_s/ω of the unsteady motion but, unless the separation bubble is very small, will eventually grow to be $O(U_s/\omega)$. It is necessary to obtain a rough estimate of the characteristic streamwise dimension at which this occurs in order to determine an appropriate form of the asymptotic expansion of the eigensolution. We refer to the region that surrounds the separation point and has this characteristic dimension as the 'outer region'.

Taking some liberty with the relatively unimportant case of Brillouin point separation, we define the scale of the outer region to be the streamwise distance from the separation point where the displacement of the separation streamline is $O(U_s/\omega)$, i.e. where

$$H_s = O(1).$$

This occurs at small values of $\bar{x}$ when the characteristic transverse dimension ld of the body is large compared with U_s/ω, and occurs at $\bar{x} = O(1)$ when ld is $O(U_s/\omega)$ (see figure 2).

For small $\bar{z}$ (Imai 1953; Sychev 1972; Smith 1977; Birkhoff & Zarantonello 1957, pp. 139, 140; Milne-Thomson 1960, pp. 336–340)

$$\bar{H}_s = \bar{a}\,\bar{x}^{\frac{3}{2}}+\bar{b}\,\bar{x}^2+\bar{c}\,\bar{x}^{\frac{5}{2}}+O(\bar{x}^3) \quad \text{as} \quad \bar{x}\to 0, \tag{3.6}$$

$$\bar{H}_b = \bar{b}\,\bar{x}^2+O(\bar{x}^3) \quad \text{as} \quad \bar{x}\to 0, \tag{3.7}$$

$$\zeta_0 \equiv U-\mathrm{j}V = 1-\frac{3\mathrm{j}\bar{a}}{2}\bar{z}^{\frac{1}{2}}-(\tfrac{15}{8}\bar{a}^2+\mathrm{j}2\bar{b})\,\bar{z}-\left[\tfrac{23}{4}\bar{a}\,\bar{b}+\mathrm{j}\left(\frac{5\bar{c}}{2}-\tfrac{15}{4}\bar{a}^3\right)\right]\bar{z}^{\frac{3}{2}}+O(\bar{z}^2) \quad \text{as} \quad \bar{z}\to 0, \tag{3.8}$$

where $\bar{a}$ is a non-negative real constant, $\bar{b}$ and $\bar{c}$ are real constants, and we have taken the real axis to be tangent to the body surface at the separation point $\bar{z} = 0$.

The separation point is a Brillouin point if and only if $\bar{a} = 0$ (Imai 1953; Sychev 1971; Birkhoff & Zarantonello 1957, pp. 139, 140). Since the free-streamline and body-surface curvatures are equal to the second derivatives of $\bar{H}_s$ and $\bar{H}_b$ respectively, (3.6) and (3.7) imply that these curvatures becomes equal to each other when separation occurs at a Brillouin point and that the free streamline curvature is otherwise infinite at separation.

Let $x_1 = \delta x$ be $O(1)$ in the outer region so that δ measures the scale of this region. Three cases can occur.

Case (1) $d \gg \epsilon$, $\bar{a} = 0$. It follows from (3.6) and (3.7) that

$$H_s \approx \frac{1}{\epsilon}\bar{b}\left(\frac{\epsilon}{\delta}\right)^2 x_1^2$$

will be $O(1)$ when $x_1 = O(1)$ if we put

$$\delta = \epsilon^{\frac{1}{2}}, \quad \bar{a} = 0. \tag{3.9}$$

It then follows from (3.8) that

$$\zeta_0 = 1 + O(\delta).$$

Case (2) $d \gg \epsilon$, $\bar{a} > 0$. In this case

$$H_s \simeq \frac{1}{\epsilon}\bar{a}\frac{\epsilon^{\frac{3}{2}}}{\delta^{\frac{3}{2}}}x_1^{\frac{3}{2}}$$

will be $O(1)$ when $x_1 = O(1)$ if we put

$$\delta = \epsilon^{\frac{1}{3}}; \quad \bar{a} \neq 0, \tag{3.10}$$

and it follows from (3.8) that

$$\zeta_0 = 1 + O(\delta).$$

Case (3) $d = O(\epsilon)$. It now follows from thin-airfoil theory that $\bar{H}_s$, $\bar{H}_b = O(\epsilon)$ and $\zeta_0 = 1 + O(\epsilon)$ when $\bar{x} = O(1)$. Hence in this case we take

$$\delta = \epsilon \quad \text{for} \quad d = O(\epsilon). \tag{3.11}$$

In all cases then we can write

$$y = H_s = H_s(x_1) \quad \text{on } s, \tag{3.12}$$

$$y = H_b = H_b(x_1) \quad \text{on the solid surface}, \tag{3.13}$$

$$U = 1 + \delta U_1(x_1, \delta y) + \ldots, \tag{3.14}$$

$$V = \delta V_1(x_1, \delta y) + \ldots, \tag{3.15}$$

where H_s, H_b, U_1 and V_1 are all $O(1)$ when x_1 and δy are $O(1)$.

This suggests that the 'outer' solution will possess an expansion of the form

$$\phi_{1,2} = \phi_{1,2}^{(0)}(x, y; x_1) + \delta\phi_{1,2}^{(1)}(x, y; x_1) + \ldots. \tag{3.16}$$

4. The instability wave

The homogeneous solution is constructed in this section by first considering the outer region where $x_1 = O(1)$. We do so because this is the smallest region over which the homogeneous solution exhibits the characteristics of a Kelvin–Helmholtz instability wave. And since we are only concerned with the generation of this wave and not with its subsequent evolution, we do not have to consider the flow on any larger scale. But, as already indicated, the Kelvin–Helmholtz solution of §4.1 becomes invalid in a small region about the separation point. Since this is just the region where the instability wave and forced solution are coupled it is necessary to construct a new 'inner solution' for this region that matches onto the Kelvin–Helmholtz solution in some intermediate or overlap domain. This is done in §4.3.

4.1. *The outer solution*

We first consider the 'outer region' where $x_1 = O(1)$. Substituting (3.12) to (3.16) into (2.1)–(2.6) and (2.16)–(2.18), and equating to zero the coefficients of δ^0 we obtain

$$\nabla^2 \phi_{1,2}^{(0)} = 0, \tag{4.1}$$

$$\left.\begin{aligned} \left(1+\mathrm{i}\frac{\partial}{\partial x}\right)\mathrm{D}\phi_2^{(0)} &= \mathrm{D}\phi_1^{(0)}, \\ \left(1+\mathrm{i}\frac{\partial}{\partial x}\right)\phi_1^{(0)} &= \phi_2^{(0)} \end{aligned}\right\} \quad (y = H_s), \tag{4.2}$$

$$\mathrm{D}\phi_{1,2}^{(0)} = 0 \quad (y = H_b), \tag{4.3}$$

where

$$\mathrm{D} \equiv \frac{\partial}{\partial y}, \tag{4.4}$$

which is just the boundary-value problem for the Kelvin–Helmholtz instability wave (Drazin & Reid 1981, pp. 14 ff.) growing on a vortex sheet parallel to a plane surface. The effects of the diverging mean flow appear at the next order, but produce 'secular' terms which cause the first-order solution to become larger than the zeroth when the instability wave has propagated over a distance $O(\delta^{-1})$ on the scale of x. We avoid this breakdown of the expansion in the usual way by using the method of multiple scales (Nayfeh 1973, pp. 228, 303) to modify the zeroth-order solution, which now becomes

$$\phi_{1,2}^{(0)} = C_0\, A(x_1)\, F_{1,2}(y, x_1) \exp\left(\mathrm{i}\int_0^x \alpha(x_1)\,\mathrm{d}x\right), \tag{4.5}$$

where

$$F_1 = \mathrm{e}^{-\alpha(y-H_s)}, \tag{4.6}$$

$$F_2 = (1-\alpha)\frac{\cosh \alpha(y-H_b)}{\cosh \alpha\Delta}, \tag{4.7}$$

$$\Delta \equiv H_s - H_b$$

C_0 is a constant, the eigenvalue α is determined by the characteristic equation

$$(1-\alpha)^2 \tanh \alpha\Delta = -1, \tag{4.8}$$

and the slowly varying amplitude function $A(x_1)$ is determined by the requirement that the first-order problem, i.e. the problem for $\phi_{1,2}^{(1)}$ possesses a solution of the form

$$\phi_{1,2}^{(1)} = C_0\, F_{1,2}^{(1)}(y; x_1) \exp\left(\mathrm{i}\int_0^x \alpha(x_1)\,\mathrm{d}x\right), \tag{4.9}$$

so that it will not dominate the zeroth-order solution (4.6) when $x = O(\delta^{-1})$.

Again substituting (3.12)–(3.16) into (2.1)–(2.6) and (2.16)–(2.18), but now equating to zero the coefficients of δ and using (4.5) to eliminate $\phi^{(0)}_{1,2}$, we find that $\phi^{(1)}_{1,2}$ is determined by

$$\nabla^2\phi^{(1)}_{1,2} = -\mathrm{i}C_0\left[\alpha' AF_{1,2}+2\alpha\frac{\partial}{\partial x_1}(AF_{1,2})\right]\exp\left(\mathrm{i}\int_0^x \alpha\,\mathrm{d}x\right), \tag{4.10}$$

$$\left(1+\mathrm{i}\frac{\partial}{\partial x}\right)\mathrm{D}\phi^{(1)}_2-\mathrm{D}\phi^{(1)}_1$$
$$= -\mathrm{i}C_0\left\{\frac{\partial}{\partial x_1}(A\,\mathrm{D}F_2)+H'_{\mathrm{s}}[\alpha(F_1-F_2)+2\alpha^2F_2]\,A\right\}\exp\left(\mathrm{i}\int_0^x \alpha\,\mathrm{d}x\right), \tag{4.11}$$

$$\left(1+\mathrm{i}\frac{\partial}{\partial x}\right)\phi^{(1)}_1-\phi^{(1)}_2 = -\mathrm{i}C_0\left[\frac{\partial}{\partial x_1}(AF_1)-H'_{\mathrm{s}}\alpha AF_1\right]\exp\left(\mathrm{i}\int_0^x \alpha\,\mathrm{d}x\right) \tag{4.12}$$

for $y = H_{\mathrm{s}}$, and

$$\mathrm{D}\phi^{(1)}_2 = \mathrm{i}C_0\,\alpha H'_{\mathrm{b}}AF_2\exp\left(\mathrm{i}\int_0^x \alpha\,\mathrm{d}x\right) \tag{4.13}$$

for $y = H_{\mathrm{b}}$, where the primes denote total derivatives with respect to x_1.

Inserting (4.9) into (4.10), dividing by $C_0\exp\mathrm{i}\int\alpha\,\mathrm{d}x)$, multiplying by $F_{1,2}$, integrating by parts over $H_{\mathrm{s}}\leqslant y\leqslant\infty$ and $H_{\mathrm{b}}\leqslant y\leqslant H_{\mathrm{s}}$ and using the fact (which follows from (4.1) and (4.5)) that $\mathrm{D}^2F_{1,2}-\alpha^2F_{1,2}=0$, we obtain

$$\frac{\mathrm{i}}{A}\int_{H_{\mathrm{s}}}^{\infty}\frac{\partial}{\partial x_1}(\alpha A^2F_1^2)\,\mathrm{d}y = (F_1\,\mathrm{D}F^{(1)}_1-F^{(1)}_1\,\mathrm{D}F_1)|_{y=H_{\mathrm{s}}}, \tag{4.14}$$

$$\frac{\mathrm{i}}{A}\int_{H_{\mathrm{b}}}^{H_{\mathrm{s}}}\frac{\partial}{\partial x_1}(\alpha A^2F_2^2)\,\mathrm{d}y = F_2\,\mathrm{D}F^{(1)}_2|_{y=H_{\mathrm{b}}}-(F_2\,\mathrm{D}F^{(1)}_2-F^{(1)}_2\,\mathrm{D}F_2)|_{y=H_{\mathrm{s}}}. \tag{4.15}$$

Inserting (4.6), (4.7) and (4.9) into (4.11)–(4.13), we obtain

$$(1-\alpha)\,\mathrm{D}F^{(1)}_2-\,\mathrm{D}F^{(1)}_1 = -\mathrm{i}\left[\frac{\partial}{\partial x_1}(A\mathrm{D}F_2)+H'_{\mathrm{s}}A\alpha^2(3-2\alpha)\right], \tag{4.16}$$

$$(1-\alpha)\,F^{(1)}_1-F^{(1)}_2 = -\mathrm{i}\left[\frac{\partial}{\partial x_1}(AF_1)-H'_{\mathrm{s}}\alpha A\right] \tag{4.17}$$

for $y = H_{\mathrm{s}}$, and

$$\mathrm{D}F^{(1)}_2 = \frac{\mathrm{i}\alpha(1-\alpha)}{\cosh\alpha\Delta}H'_{\mathrm{b}}A \tag{4.18}$$

for $y = H_{\mathrm{b}}$.

Adding (4.14) and (4.15) using (4.16)–(4.18) to eliminate $F^{(1)}_{1,2}$, inserting (4.6) and (4.7) and carrying out the integrations, we obtain after considerable manipulation which takes advantage of (4.8)

$$\left[A^2\left(\frac{\alpha\Delta}{2\sinh\alpha\Delta\cosh\alpha\Delta}+\frac{\alpha}{\alpha-1}\right)\right]' = 0. \tag{4.19}$$

Since differentiating (4.8) with respect to Δ yields

$$\frac{\alpha\Delta}{2\sinh\alpha\Delta\cosh\alpha\Delta}\left(1+\frac{\alpha}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}\alpha}\right) = \frac{\alpha}{1-\alpha},$$

(4.19) can be written as

$$\left(\frac{A^2\alpha^2}{\sinh 2\alpha\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}\alpha}\right)' = 0,$$

which upon integration becomes

$$A^2 = \frac{3 \sinh 2\alpha\Delta}{2\alpha^2} \frac{d\alpha}{d\Delta},$$

where we have set the irrelevant constant of integration equal to $\frac{3}{2}$. Finally using (4.8) we obtain the remarkably simple result

$$A = -\frac{\cosh \alpha\Delta}{\alpha(1-\alpha)} \left(3 \frac{d\alpha}{d\Delta}\right)^{\frac{1}{2}}. \tag{4.20}$$

Except for the multiplicative constant C_0, the lowest-order outer solution (4.5) is now *completely* determined. C_0 is the 'coupling coefficient' alluded to in §1.

4.2. *The inner expansion of the outer solution*

We now consider the limiting form of this solution as $x_1 \to 0$. Since $\Delta \to 0$ in this limit, it follows from (4.8) that

$$\alpha^3\Delta \to -1 \quad \text{as} \quad x_1 \to 0,$$

or, choosing the root corresponding to the spatially growing wave,

$$\alpha \to e^{-\frac{1}{3}i\pi}/\Delta^{\frac{1}{3}} \tag{4.21}$$

and

$$\frac{d\alpha}{d\Delta} \to -\frac{\alpha}{3\Delta} \tag{4.22}$$

as $x_1 \to 0$.

Then since $\Delta \equiv H_s - H_b$ and $\bar{x}$ is certainly small when x_1 is, it follows from (3.4)–(3.7), (3.9)–(3.11) and (4.20) that

$$\Delta \to \bar{a}\epsilon^{\frac{1}{2}}\bar{x}^{\frac{3}{2}} = a x_1^{\frac{3}{2}} \quad \text{for} \quad \bar{a} \neq 0, \tag{4.23}$$

$$\Delta \to \bar{c}\epsilon^{\frac{3}{2}}\bar{x}^{\frac{5}{2}} = \delta^{\kappa} c x_1^{\frac{5}{2}} \quad \text{for} \quad \bar{a} = 0, \tag{4.24}$$

where $\kappa = \frac{1}{2}$, 0 for $d = O(1)$, $O(\epsilon)$, and

$$A \to 1 \quad \text{as} \quad x_1 \to 0, \tag{4.25}$$

where we have put

$$a, \quad b, \quad c = \begin{cases} \bar{a}, \quad \bar{b}, \quad \bar{c} \quad \text{for} \quad d = O(1), \\ \dfrac{\bar{a}}{\epsilon}, \quad \dfrac{\bar{b}}{\epsilon}, \quad \dfrac{\bar{c}}{\epsilon} \quad \text{for} \quad d = O(\epsilon) \end{cases} \tag{4.26}$$

so that, with one possible exception, the real constants a, b and c are independent of ϵ in all cases. The exception occurs because, as indicated in §7, $\bar{a}$ actually scales with the inverse Reynolds number to the $\frac{1}{16}$ power, independently of ϵ, for the case of slender body laminar separation. But this causes no difficulty for our purposes and the reader can, for consistency, always suppose the ϵ is of the order of the $-\frac{1}{16}$ power of the Reynolds number in this case.

Hence it follows from (2.2), (3.16), (4.5) and (4.6)

$$\boldsymbol{u}_1 \to \{i, -1\} \frac{C_0 \lambda}{r\tilde{\delta}^{r-1}} \left(\frac{\tilde{\delta}^{r-1}}{x}\right)^{1-1/r} \exp\left[\lambda \left(\frac{x}{\tilde{\delta}^{r-1}}\right)^{1/r} \left(i - \frac{y}{rx}\right) - it\right] \quad \text{as} \quad x_1 \to 0, \tag{4.27}$$

where

$$\tilde{\delta} = \delta, \quad r = 2, \quad \lambda = \frac{2e^{-\frac{1}{3}i\pi}}{a^{\frac{1}{3}}} \quad \text{if} \quad a > 0, \tag{4.28}$$

$$\tilde{\delta} = \begin{cases} \delta^{\frac{6}{5}} & \text{for} \quad d = O(1), \\ \delta & \text{for} \quad d = O(\epsilon), \end{cases} \quad r = 6, \quad \lambda = \frac{6e^{-\frac{1}{3}i\pi}}{c^{\frac{1}{3}}} \quad \text{if} \quad a = 0. \tag{4.29}$$

As already indicated, this solution becomes invalid in a small semicircular region surrounding the separation point $x, y = 0$. It does not, for example, have zero normal velocity on the solid surface just upstream of this point as required by (2.18) and (3.7) and it is not even valid in the downstream region where $x > 0$ and $x = O(y)$. Thus (4.10) and (4.21)–(4.23) imply that

$$\phi_1^{(1)} \sim -\frac{\mathrm{i}y}{4\delta x}\phi_1^{(0)} + \dots \quad \text{as} \quad x_1 \to 0 \quad \text{with} \quad x = O(y)$$

when $a > 0$, so that the first-order term becomes as large as the zeroth-order term and the assumed expansion (3.16) breaks down.

4.3. *The inner solution*

However, the form of (4.27) suggests that we can obtain the solution in the region surrounding the separation point by introducing the new 'inner variables'

$$\{x_2, y_2\} = \left\{\frac{x}{\delta^{r-1}}, \frac{y}{\delta^{r-1}}\right\} \tag{4.30}$$

into the governing equations. For simplicity, we only consider the case where separation does not occur at a Brillouin point (i.e. the case where $\bar{a} > 0$).

Since (3.5), (3.7), (3.10), (3.11), and (4.26) imply that

$$H_\mathrm{b} = \delta^2 b x^2 \quad \text{for} \quad d = O(\epsilon) \tag{4.31}$$

and

$$H_\mathrm{b} = \delta^3 b x^2 \quad \text{for} \quad d = O(1), \tag{4.32}$$

and since $y_2 \ll x_2$ in region 2, it follows from (2.4)–(2.6) and (2.18) that the solutions in this region will be of the form

$$p_2 = \delta \bar{p}_2^{(0)}(x_2, t) + \dots, \tag{4.33}$$

$$u_2 = \bar{u}_2^{(0)}(x_2, t) + \dots, \tag{4.34}$$

$$v_2 = -y_2 \frac{\partial}{\partial x_2}\bar{u}_2^{(0)}(x_2, t) + O(\delta^3), \quad x_2 > 0, \tag{4.35}$$

where, as indicated by the arguments, $\bar{p}_0^{(0)}$ and $\bar{u}_2^{(0)}$ are independent of y_2.

The form of the free-surface boundary condition suggests that the solution in region 1 will have an expansion of the form

$$p_1 = \delta \bar{p}_1^{(0)}(x_2, y_2, t) + \dots, \tag{4.36}$$

$$u_1 = \delta \bar{u}_1^{(0)}(x_2, y_2, t) + \dots, \tag{4.37}$$

$$v_1 = \delta \bar{v}_1^{(0)}(x_2, y_2, t) + \dots. \tag{4.38}$$

Since (3.4), (3.6), (3.8), (3.10), (3.11) and (4.26) imply

$$H_\mathrm{s} = a\delta^{\frac{3}{2}} x^{\frac{3}{2}} + \dots, \tag{4.39}$$

$$U = 1 + \tfrac{3}{4}\delta^{\frac{3}{2}} a \frac{y}{x^{\frac{1}{2}}} + \dots, \tag{4.40}$$

$$V = \tfrac{3}{2} a \delta^{\frac{3}{2}} x^{\frac{1}{2}} + \dots, \tag{4.41}$$

provided that $y \ll x$. Substituting (4.30) and (4.33) to (4.38) into (2.16) to (2.18) yields

$$\frac{\partial \bar{v}_1^{(0)}}{\partial t} = -a x_2^{\frac{1}{2}} \left(\frac{3}{4} \frac{\bar{u}_2^{(0)}}{x_2} + 3 \frac{\mathrm{d}\bar{u}_2^{(0)}}{\mathrm{d}x_2} + x_2 \frac{\mathrm{d}^2 \bar{u}_2^{(0)}}{\mathrm{d}x_2^2} \right) \quad \text{for} \quad x_2 > 0, \tag{4.42}$$

$$\bar{p}_1^{(0)} = \bar{p}_2^{(0)}(x_2, t) \quad \text{for} \quad x_2 > 0, \tag{4.43}$$

$$\bar{v}_1^{(0)} = 0 \quad \text{for} \quad x_2 < 0, \tag{4.44}$$

where, to the order of approximation of the analysis, we can suppose that the boundary conditions are imposed at $y_2 = 0$ and we have used total derivatives to indicate that the quantities are independent of y_2, even though they still depend on t. Equations (2.1)–(2.5) imply that

$$\bar{p}_1^{(0)}(x_2, y_2, t) = -\bar{u}_1^{(0)}(x_2, y_2, t), \tag{4.45}$$

$$\frac{\mathrm{d}\bar{p}_2^{(0)}}{\mathrm{d}x_2} = -\frac{\partial}{\partial t} \bar{u}_2^{(0)}(x_2, t), \tag{4.46}$$

and that

$$\bar{w}_1 \equiv \bar{u}_1^{(0)} - \mathrm{j}\bar{v}_1^{(0)} \tag{4.47}$$

is an analytic function of the complex variable

$$z_2 \equiv x_2 + \mathrm{j}y_2 \tag{4.48}$$

Eliminating $\bar{p}_1^{(0)}$, $\bar{p}_2^{(0)}$ and $\bar{u}_2^{(0)}$ between (4.42), (4.43), (4.45) and (4.46), using (2.2) to eliminate the derivative with respect to time and inserting (4.28), we obtain

$$x_2^{\frac{3}{2}} \frac{\partial^3 \bar{u}_1^{(0)}}{\partial x_2^3} + 3x_2^{\frac{1}{2}} \frac{\partial^2 \bar{u}_1^{(0)}}{\partial x_2^2} + \frac{3}{4x_2^{\frac{1}{2}}} \frac{\partial \bar{u}_1^{(0)}}{\partial x_2} + \left(\frac{\lambda}{2}\right)^3 \bar{v}_1^{(0)} = 0 \quad \text{for} \quad y_2 = 0, \quad x_2 > 0. \tag{4.49}$$

This boundary condition will clearly be satisfied if the analytic function $\bar{w}_1$ introduced in (4.47) and (4.48) satisfies the ordinary differential equation

$$z_2^{\frac{3}{2}} \frac{\mathrm{d}^3 \bar{w}_1}{\mathrm{d}z_2^3} + 3z_2^{\frac{1}{2}} \frac{\mathrm{d}^2 \bar{w}_1}{\mathrm{d}z_2^2} + \frac{3}{4z_2^{\frac{1}{2}}} \frac{\mathrm{d}\bar{w}_1}{\mathrm{d}z_2} + \mathrm{j}\left(\frac{\lambda}{2}\right)^3 \bar{w}_1 = 0, \tag{4.50}$$

and if we can obtain a solution that also satisfies (4.44), we will have succeeded in solving the complete inner problem.

In order to solve (4.50), we introduce the new independent variable

$$T \equiv \xi + \mathrm{j}\eta = z_2^{\frac{1}{2}} \tag{4.51}$$

to obtain

$$\frac{\mathrm{d}^3 \bar{w}_1}{\mathrm{d}T^3} + \frac{3}{T} \frac{\mathrm{d}^2 \bar{w}_1}{\mathrm{d}T^2} + \mathrm{j}\lambda^3 \bar{w}_1 = 0, \tag{4.52}$$

which possesses the solution

$$\bar{w}_1 = c_0 \frac{\mathrm{j}}{T} \mathrm{e}^{\mathrm{j}\lambda T - \mathrm{i}t}, \tag{4.53}$$

where c_0 is an arbitrary constant. Or in terms of real quantities

$$\bar{u}_1^{(0)} = c_0 \frac{\eta \cos \lambda\xi - \xi \sin \lambda\xi}{\eta^2 + \xi^2} \mathrm{e}^{-\lambda\eta - \mathrm{i}t}, \tag{4.54}$$

$$\bar{v}_1^{(0)} = -c_0 \frac{\eta \sin \lambda\xi + \xi \cos \lambda\xi}{\eta^2 + \xi^2} \mathrm{e}^{-\lambda\eta - \mathrm{i}t}. \tag{4.55}$$

Equation (4.51) implies that $\xi = 0$ when $y_2 = 0$ and $x_2 < 0$. Hence it follows from

(4.55) that $\bar{v}_1^{(0)}$ vanishes and the boundary condition (4.44) is indeed satisfied. Equations (4.54) and (4.55) therefore provide an acceptable lowest-order inner solution. We must now show that they can be 'matched' onto the outer instability-wave solution (4.5), i.e. that they can be made to coincide with its inner expansion (4.27) (see (4.28)) in some overlap domain.

4.4. *Matching inner and outer expansions*

To this end note that (4.51) implies

$$\xi \to x_2^{\frac{1}{2}}, \quad \eta \to \frac{y_2}{2x_2^{\frac{1}{2}}} \quad \text{as} \quad x_2 \to +\infty \quad \text{with } y_2 \text{ fixed.} \tag{4.56}$$

Hence it follows from (4.28), (4.37), (4.38), (4.54) and (4.55) that

$$\{u_1, v_1\} \to \{\mathrm{i}, -1\}\frac{\delta c_0}{2x_2^{\frac{1}{2}}}\exp\left[\lambda\left(\mathrm{i}x_2^{\frac{1}{2}} - \frac{y_2}{2x_2^{\frac{1}{2}}}\right) - \mathrm{i}t\right] \quad \text{as} \quad x_2 \to \infty. \tag{4.57}$$

Equations (4.28) and (4.30) now show that this will coincide with (4.27) it we take

$$c_0 = C_0\lambda/\delta^2. \tag{4.58}$$

This completes the solution for the instability wave.

5. The forced solution

In this section we construct a forced or particular solution due to a harmonic point source at

$$\{x, y\} = \{x_0, y_0\}, \quad x_0^2 + y_0^2 = O(1). \tag{5.1}$$

We suppose that the result does *not* involve a Kelvin–Helmholtz instability wave and therefore that it remains bounded at downstream infinity. We again consider only the case where the separation point is not a Brillouin point.

Equation (5.1) implies that the dimensional distance between the source and separation points is $O(U_\infty/\omega)$, i.e. that the source is located within several wavelengths of the separation point. This simplifies the analysis without significantly affecting the coupling between the forcing and the eigensolution of §4, which is, of course, the primary issue of this work. In fact, we subsequently show that this coupling is substantially independent of the precise nature and location of the source.

In §5.2 we show that the present result, unlike the eigensolution of §4, is a uniformly valid solution of the linearized inviscid equations in any neighbourhood of the separation point.

5.1. *Construction of solution*

We begin by finding the appropriate form of the boundary and jump conditions. Since $\bar{x}, \bar{y} = O(1)$ is small on the scale of (5.1), H_b, H_s, U and V are again given by (4.31) or (4.32) and (4.39)–(4.41) respectively.

Substituting these into (2.16)–(2.18), we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)v_2 - \frac{\partial}{\partial t}v_1 = \tfrac{3}{2}a\delta^{\frac{3}{2}}x^{\frac{1}{2}}\left[\frac{\partial u_2}{\partial x} - \frac{\partial v_2}{\partial y} + \frac{\partial}{\partial t}(u_2 - u_1) + \frac{u_2}{2x}\right] + O(\delta^2), \tag{5.2}$$

$$\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t} = \tfrac{3}{2}a\delta^{\frac{3}{2}}\frac{v_2}{2x^{\frac{1}{2}}} + O(\delta^2) \tag{5.3}$$

for $y = \delta^{\frac{3}{2}}ax^{\frac{3}{2}}$, $x > 0$ and

$$v_{1,2} = O(\delta^2) \quad \text{for} \quad y = 0, \quad x \lessgtr 0. \tag{5.4}$$

Transferring (5.2) and (5.3) to $y = 0$ by expanding in a Taylor series, using (2.2), (2.3), (2.5) and (2.6) to eliminate $\partial v_{1,2}/\partial y$ and (5.4) to eliminate v_2 and $\partial v_2/\partial x$ from the result, we obtain

$$\frac{\partial v_1}{\partial t} = -a\delta^{\frac{3}{2}}x^{\frac{1}{2}}\left[\frac{3u_2}{4x} + 3\frac{\partial u_2}{\partial x} + \frac{3}{2}\frac{\partial}{\partial t}(u_2 - u_1) + x\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\frac{\partial u_2}{\partial x} - x\frac{\partial^2 u_1}{\partial x\,\partial t}\right] + O(\delta^2)$$
$$\text{for} \quad y = 0, \quad x > 0, \tag{5.5}$$

and

$$\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t} = -a\delta^{\frac{3}{2}}x^{\frac{3}{2}}\frac{\partial^2 p_1}{\partial t\,\partial y} + O(\delta^2) \quad \text{for} \quad y = 0, \quad x > 0. \tag{5.6}$$

Equations (5.4)–(5.6) suggest that the solution will have an expansion of the form

$$\phi_{1,2} = \phi^{(0)}_{1,2}(x,y) + \delta^{\frac{3}{2}}\phi^{(1)}_{1,2}(x,y) + \dots \tag{5.7}$$

and that $p_{1,2}$, $u_{1,2}$ and $v_{1,2}$ will possess similar expansions with $p^{(0,1)}_{1,2}$, $u^{(0,1)}_{1,2}$ and $v^{(0,1)}_{1,2}$ determined from the corresponding $\phi^{(0,1)}_{1,2}$ by (2.1), (2.2), (2.4) and (2.6). Notice that we are using the same notation as was used in (3.6) for the homogeneous solution, but this should cause no confusion.

For definiteness, we choose the source to be such that

$$\phi_1 \to \ln\left[(x-x_0)^2 + (y-y_0)^2\right]^{\frac{1}{2}} \quad \text{as} \quad x, y \to x_0, y_0. \tag{5.8}$$

Then it follows from (5.4)–(5.6) and (2.1), (2.2), (2.4) and (2.5) that

$$\phi^{(0)}_1 = \ln\left[(x-x_0)^2 + (y-y_0)^2\right]^{\frac{1}{2}} + \ln\left[(x-x_0)^2 + (y+y_0)^3\right]^{\frac{1}{2}} \tag{5.9}$$

and that

$$-\frac{\partial u^{(0)}_2}{\partial t} = P_0'(x,t) \equiv \frac{\partial p^{(0)}_1(x,0,t)}{\partial x}. \tag{5.10}$$

Thus the zeroth-order solution is just the solution for a point source near a doubly infinite plane wall, and $P_0'(x)$ is just the streamwise surface pressure gradient produced by that source.

Substituting (5.7) into (2.1) and (2.2), using the result together with (5.9) in (5.4) and (5.5), we obtain, upon subtracting out the zeroth-order solution and using (4.28),

$$v^{(1)}_1 = 0 \quad \text{for} \quad y = 0, \quad x < 0, \tag{5.11}$$

$$v^{(1)}_1 = \left(\frac{2}{\lambda}\right)^3\left[x^{\frac{1}{2}}\frac{3}{4x}P_0' - H(x)\right] \quad \text{for} \quad y = 0, \quad x > 0, \tag{5.12}$$

$$H \equiv \left[\left(3 + x\frac{\mathrm{d}}{\mathrm{d}x}\right)\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\left(\tfrac{9}{2} + 2x\frac{\mathrm{d}}{\mathrm{d}x}\right)\right]\frac{\mathrm{d}}{\mathrm{d}x}u^{(0)}_1. \tag{5.13}$$

Since (2.2) and (2.3) imply that $u^{(1)}_1 - \mathrm{j}v^{(1)}_1$ is an analytic function of $z = x + \mathrm{j}y$, it follows from (5.11), (5.12) and the theory of analytic functions that

$$u^{(1)}_1 - \mathrm{j}v^{(1)}_1 = -\frac{1}{\pi}\left(\frac{2}{\lambda}\right)^3\int_0^\infty \frac{\frac{3}{4}\tilde{x}^{-\frac{1}{2}}P_0'(\tilde{x}) - \tilde{x}^{\frac{1}{2}}H(\tilde{x})}{\tilde{x} - z}\,\mathrm{d}\tilde{x} \quad \text{for} \quad y > 0. \tag{5.14}$$

5.2. *The inner expansion*

Near $z = 0$ (5.14) behaves like (Gakhov 1966, pp. 55, 56)

$$u_1^{(1)} - \mathrm{j}v_1^{(1)} = -\mathrm{j}\left(\frac{2}{\lambda}\right)^3 \frac{3}{4}\frac{P_0'(0,t)}{z^{\frac{1}{2}}} + \dots. \tag{5.15}$$

Hence it follows from (2.2), (5.7) and (5.9) that the complete solution behaves like

$$u_1 - \mathrm{j}v_1 = -2\frac{x_0}{r_0^2}(1+\mathrm{i}z)\,\mathrm{e}^{-\mathrm{i}t} - P_0'(0)\left(\frac{6}{\lambda^3}\frac{\delta^{\frac{3}{2}}\mathrm{j}}{z^{\frac{1}{2}}} + z\right) + \dots, \tag{5.16}$$

where we have put

$$r_0 \equiv (x_0^2 + y_0^2)^{\frac{1}{2}}. \tag{5.17}$$

Introducing the inner variable T, defined by (4.30), (4.48) and (4.51), we find that

$$u_1 - \mathrm{j}v_1 = -\frac{2x_0}{r_0^2}\mathrm{e}^{-\mathrm{i}t} + \delta\left[\frac{2x_0}{\mathrm{i}r_0^2}T^2\,\mathrm{e}^{-\mathrm{i}t} - P_0'(0)\left(\frac{6\mathrm{j}}{\lambda^3 T} + T^2\right)\right] \tag{5.18}$$

when $T = O(1)$.

By introducing the inner variables x_2 and y_2 into the boundary conditions (5.4)–(5.6) it is easy to show that (5.18) represents the correct expansion of the forced solution in the 'inner' region $x_2, y_2 = O(1)$, i.e. the expansion (5.7) is uniformly valid in both the inner and outer regions to the order of approximation of the analysis.

6. Determination of amplitude of instability wave – the receptivity problem

As noted in §1, both the forced solution (5.19) and the instability-wave eigensolution (4.53) possess square-root singularities at the separation point $x, y = 0$. We can now eliminate the singularity, i.e. impose a Kutta condition, by adding these two solutions and adjusting the arbitrary constant c_0 to cancel the singularity. Thus it follows from (4.37), (4.38), (4.47), (4.51), (4.53) and (5.19) that the Kutta condition will be satisfied if we take

$$c_0\,\mathrm{e}^{-\mathrm{i}t} = \frac{6}{\lambda^3}P_0'(0,t). \tag{6.1}$$

Hence it follows from (4.28) and (4.58) that the initial amplitude C_0 of the instability wave (4.5), which we refer to as the 'coupling coefficient', is given by

$$C_0 = -\frac{3\delta^2}{4\lambda}a[P'(0,t)\,\mathrm{e}^{\mathrm{i}t}]. \tag{6.2}$$

Although, as indicated in §1, the validity of the Kutta condition is now well established for sharp trailing edges, it has not been previously proposed for flows separating from smooth surfaces. It is therefore worth providing some evidence for its validity in this case. To this end, we must first consider the more general case where this condition is not imposed. Then since both the forced solution (5.19) and the instability-wave solution (4.53) become singular at the separation point, the spatial gradients become large and cause the flow to behave in a quasi-steady – though possibly nonlinear – fashion there. There will therefore be a small region (relative to the size $l\epsilon\delta$ of the inner inviscid region) where the complete solution (steady plus unsteady) is given by a Kirchhoff free-streamline solution with its separation point displaced from the steady position by an amount $l\epsilon\alpha_0\,\mathrm{Re}\,x_s\,\mathrm{e}^{-\mathrm{i}t}$ and with its effective freestream velocity differing from the steady value of unity by an amount

$\alpha_0 \operatorname{Re} u_e \mathrm{e}^{-\mathrm{i}t}$, where Re denotes the real part with respect to i. Thus assuming that $l\epsilon\alpha_0 |x_s| \ll l\epsilon\delta$, i.e.

$$|x_s|\,\alpha_0 \ll \delta, \tag{6.3}$$

the solution in this region is given by

$$\zeta_1 = 1+\operatorname{Re}(\alpha_0 u_e \mathrm{e}^{-\mathrm{i}t}) - \frac{3\mathrm{j}}{2}(\bar{a}+\alpha_0 \operatorname{Re}\bar{a}_1 \mathrm{e}^{-\mathrm{i}t})(\bar{z}-\epsilon\alpha_0 \operatorname{Re} x_s \mathrm{e}^{-\mathrm{i}t})^{\frac{1}{2}}+\dots \tag{6.4}$$

rather than by the appropriate approximation to the steady Kirchhoff solution (3.8) plus an unsteady perturbation.

Equation (6.4) implies that $\mathrm{d}\zeta/\mathrm{d}\bar{z}$ becomes infinite at the separation point, and consequently insures that the convective terms will dominate the time-derivative term in the Euler equations. This suggests that the (nonlinear) quasi-steady approximation (6.4) will indeed satisfy the relevant inviscid equations in the vicinity of the separation point.

In fact, it can be shown by direct substitution that these equations are satisfied by

$$\zeta_1 = 1+\operatorname{Re}(\alpha_0 u_e \mathrm{e}^{-\mathrm{i}t}) - \frac{3\mathrm{j}}{2}\left(a+\frac{1}{\lambda\delta}\alpha_0 \operatorname{Re} a_1 \mathrm{e}^{-\mathrm{i}t}\right)\delta^{\frac{3}{2}} z_0^{\frac{1}{2}}$$

$$+\alpha_0 \operatorname{Re}[(\mathrm{i}u_e - x_s)\mathrm{e}^{-\mathrm{i}t}]z_0 + O(z_0^{\frac{3}{2}}), \tag{6.5}$$

$$\zeta_2 = -\operatorname{Re}\mathrm{i}\alpha_0[z_s \mathrm{e}^{-\mathrm{i}t}+O(z_0^{\frac{1}{2}})], \tag{6.6}$$

$$H_s = \delta^{\frac{3}{2}}\left[a+\alpha_0 \operatorname{Re}\left\{\frac{a_1}{\lambda\delta}-\mathrm{i}a(x_s-\mathrm{i}u_e)\right\}\mathrm{e}^{-\mathrm{i}t}\right]x_0^{\frac{3}{2}}+O(x_0^2) \quad \text{as} \quad z_0 \to 0, \tag{6.6}$$

where

$$x_0 = x-\alpha_0 \operatorname{Re} x_s \mathrm{e}^{-\mathrm{i}t}, \tag{6.7}$$

$$z_0 = x_0+\mathrm{j}y, \tag{6.8}$$

and we have put $a_1 = \lambda\delta\bar{a}_1$, $\lambda\bar{a}_1$ for $d = O(1)$, $O(\epsilon)$ in anticipation of the fact that $\bar{a}_1$ can be $O(\delta^{-1})$ as $\epsilon \to 0$, and used (3.10), (3.11) and (4.26).

It now appears that for both laminar and turbulent boundary layers and for both steady and moderately unsteady separations (Sychev 1972, 1979; Sychev & Sychev 1980; Smith 1977; Elliott, Smith & Cowley 1983) the viscous and nonlinear flow in the vicinity of the separation point is determined by a local reaction and will match onto the solution (6.5) at large distances from the reaction zone, which can always be taken as small relative to the scale $l\epsilon\delta$ of the inner inviscid region when the Reynolds number is sufficiently large relative to ϵ^{-1}. The coefficient $a+(\lambda\delta)^{-1}\alpha_0 \operatorname{Re} a_1 \mathrm{e}^{-\mathrm{i}t}$ is determined by the flow in the interaction zone, a being the steady-state value and $(\lambda\delta)^{-1}\alpha_0 \operatorname{Re} a_1 \mathrm{e}^{-\mathrm{i}t}$ being the deviation produced by the unsteady motion (see e.g. discussion of the quasi-steady laminar separation in §8). The determination of a and a_1 requires a detailed solution of the viscous flow in the interaction zone.

We assume that u_e can be expanded as

$$u_e = u_e^{(0)}+\delta u_e^{(1)}+\dots. \tag{6.9}$$

Notice, however, that $z_0^{\frac{1}{2}}$ cannot necessarily be expanded in powers of α_0 when $z = O(\alpha_0)$, as it usually will be within the interaction zone, but it can be so expanded when z is sufficiently large, as it will be in the inner inviscid region, $\bar{z} = O(\delta\epsilon)$, according to the inequality (6.3). Then in the latter region (6.5) becomes

$$\zeta_1 = \zeta_0+\alpha_0\left\{u_e+\delta\left[\frac{3\mathrm{j}a}{4}x_s\frac{1}{T}+(\mathrm{i}u_e-x_s)T^2-\tfrac{3}{2}\mathrm{j}a_1\frac{1}{\lambda}T\right]\right\}\mathrm{e}^{-\mathrm{i}t}+O(\delta^3), \tag{6.10}$$

where we have introduced the inner inviscid variable T given by (4.30), (4.48) and (4.51) and dropped the symbol Re, since the expression is now linear in e^{-it}. Here ζ_0 denotes the appropriate approximation to the steady Kirchhoff solution (3.8).

It follows from (4.53) that the instability-wave solution behaves like

$$u_1 - jv_1 = \delta c_0 \left[\frac{j}{T} - \lambda - \tfrac{1}{2}j\lambda^2 T + \tfrac{1}{6}\lambda^3 T^2 + O(T^3)\right] e^{-it} + c_0\, O(\delta^2). \tag{6.11}$$

Matching this with (5.18) and (6.10), we find upon using (4.28), (6.8) and (6.9) that

$$u_e^{(0)} = -\frac{2x_0}{r_0^2}, \quad u_e^{(1)} = \tfrac{3}{4}\lambda a[P_0'(0,0) - x_s], \tag{6.12}$$

$$a_1 = -2[x_s - P_0'(0,0)], \tag{6.13}$$

and most important that

$$c_0 = -\frac{3a}{4}[P_0'(0,0) - x_s]. \tag{6.14}$$

Hence it follows from (4.58) that the amplitude C_0 of the instability wave (4.5) is now given by

$$C_0 = -\frac{3}{4\lambda}\delta^2 a[P_0'(0,0) - x_s], \tag{6.15}$$

which reduces to the Kutta-condition result (6.2) when $|x_s| \ll P_0' = O(1)$. In §8 we shall show that this is indeed the case for quasi-steady laminar separations.

The previous analysis demonstrates that the singularity can be eliminated from the linearized inviscid solution by accounting for the nonlinear effects resulting from the motion of the separation point. The relation between the amplitude x_s of the separation point motion and the pressure-gradient amplitude P_0' produced by the forcing is determined by viscous (and non-parallel flow) effects in the vicinity of the separation point. The Kutta condition is approximately satisfied only when x_s is negligibly small relative to P_0'. We show that this is the case for laminar quasi-steady separations in §8. Notice that the instability wave is always present when x_s differs from $-P_0'$.

It now follows from (2.2), (3.16), (4.5), (4.6) and (4.20) that to lowest order of approximation the streamwise velocity fluctuation produced by the instability wave is given by

$$\tilde{u}_1 = \frac{\delta^2 3i}{4\lambda}[P_0'(0,0) - x_s]\, a \frac{\cosh \alpha\Delta}{1-\alpha}\left(3\frac{d\alpha}{d\Delta}\right)^{\frac{1}{2}} \exp\left\{-\alpha(y - H_s) + i\left[\int_0^x \alpha(x_1)\, dx - t\right]\right\}, \tag{6.16}$$

where $\Delta \equiv H_s - H_b$ is the local thickness of the separated region, and the wavenumber α is given by the dispersion relation (4.8). The latter is plotted against Δ in figure 3, which shows that it goes to the limit (4.21) for small Δ. (The curves are fairly well represented by the first two terms in the asymptotic expansion, i.e. by

$$\alpha = \frac{e^{-\frac{1}{3}i\pi}}{\Delta^{\frac{1}{3}}} + \tfrac{2}{3} + O(\Delta^{\frac{1}{3}})$$

for $\Delta < 0.1$.)

For large Δ, α goes to the unbounded vortex-sheet limit $1 - i$. The figure shows that α is fairly close to this value when $\Delta \approx 2$. Notice that the equation following (4.19) then implies that

$$\frac{\cosh \alpha\Delta}{1-\alpha}\left(\frac{d\alpha}{d\Delta}\right)^{\frac{1}{2}} \to \left(\frac{\alpha}{2(1-\alpha)}\right)^{\frac{1}{2}} \to \left(\frac{1-i}{2i}\right)^{\frac{1}{2}} = \frac{1}{2^{\frac{1}{4}}} e^{-\frac{3}{8}i\pi} \quad \text{as} \quad \Delta \to \infty.$$

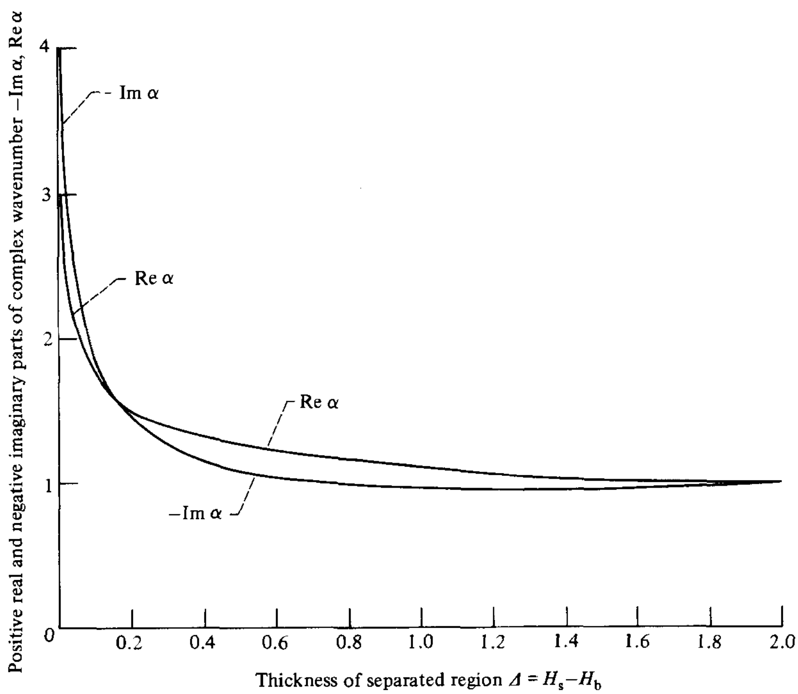

FIGURE 3. Variation of complex wavenumber with separated-region thickness.

7. Applicability of the analysis

Since the present analysis is basically inviscid and since the laminar separation point approaches a Brillouin point as the Reynolds number $Re \to \infty$ with the transverse body dimensions held fixed, it might, at first, appear that our assumption of non-Brillouin-point separation restricts the applicability of this work to turbulent boundary layers (or to thin bodies at small angles of attack). But all real flows are viscous and the coefficient $\bar{a}$ of the first term of (3.6) or (3.8) is, as shown by Sychev (1972), proportional to $Re^{-\frac{1}{16}}$ rather than being equal to zero for laminar flow. This term will then always be important for sufficiently small values of $\bar{z}$, i.e. in some region close to the separation point, and one must give very careful consideration to the conditions under which it can be neglected.

The high-Reynolds-number analyses of Smith (1977, 1979) and Sychev (1972) clearly demonstrate that the viscous effects are confined to thin shear layers and narrow boundary layers with thickness $O(Re^{-\frac{1}{2}})$ and that the flow outside these regions is inviscid and irrotational to $O(Re^{-\frac{1}{2}})$ even though it involves terms $O(Re^{-\frac{1}{16}})$ due to the displacement of the separation point through the action of viscosity. Thus the steady outer solution, which applies outside the boundary-layer, shear-layer and 'triple-deck' region, is still given to $O(Re^{-\frac{1}{2}})$ by the inviscid Kirchhoff solution (3.8), but with

$$\bar{a} = \bar{a}_0 \epsilon_0^{\frac{1}{2}}, \tag{7.1}$$

where

$$\epsilon_0 \equiv Re^{-\frac{1}{8}}, \tag{7.2}$$

and $\bar{a} = O(1)$ as $\epsilon_0 \to 0$. The flow takes on the well-known 'triple-deck' structure in a region of dimension $O(\epsilon_0^3)$ that surrounds the separation point (Sychev 1972). This is a particular example of the local interaction alluded to in §6.

If the non-Brillouin-point term $\bar{a}\bar{x}^{\frac{3}{2}}$ were dominant in the inner inviscid region, it would, in view of (7.1), be more appropriate to take $(\epsilon_0\,\epsilon)^{\frac{1}{3}}$ as the inviscid region scale for a blunt body with $d = O(1)$ (see (4.27) and (4.28)) rather than $\epsilon^{\frac{1}{3}}$, as given by (3.10) and (4.30) with $r = 2$. Thus the characteristic dimension of the inner inviscid region would then be

$$(\epsilon_0\,\epsilon)^{\frac{1}{3}}\,\epsilon l,$$

which will be large compared with the size $l\epsilon_0^3$ of the 'triple deck' if

$$S \equiv \frac{1}{\epsilon} \ll Re^{\frac{1}{4}}, \tag{7.3}$$

where S is the Strouhal number based on the characteristic streamwise body dimension l. This condition insures that there will be a region outside of the triple deck where the inner inviscid solution remains valid. Since, as indicated at the end of this section, (7.1) is independent of the slenderness ratio (ϵ in our case), (7.3) also applies for slender bodies.

It can be shown that the lowest-order inner inviscid solution for the unsteady flow is unaffected by the second term in (3.6). Then it follows from (3.2) and (7.1) that the non-Brillouin-point term will dominate in the inner inviscid region $x = O((\epsilon_0\,\epsilon)^{\frac{1}{3}})$ of the blunt body if

$$\epsilon_0^{\frac{1}{2}}\epsilon^{\frac{3}{2}}(\epsilon_0\,\epsilon)^{\frac{1}{2}} \gg \epsilon^{\frac{5}{2}}(\epsilon_0\,\epsilon)^{\frac{5}{6}} \tag{7.4}$$

or

$$S \equiv \frac{1}{\epsilon} \gg \frac{1}{\epsilon_0^{\frac{1}{8}}} = Re^{\frac{1}{64}}.$$

Thus the non-Brillouin-point term should be dominant in the inner inviscid region even for relatively small Strouhal numbers, say 2 or 3, at any Reynolds number for which the boundary layer can reasonably be expected to remain laminar. We therefore expect the present analysis to apply to blunt bodies with laminar boundary layers when†

$$Re^{\frac{1}{64}} \ll S \ll Re^{\frac{1}{4}}. \tag{7.5}$$

It is worth noting that (7.3) is also the condition for which the flow in the 'triple deck' will be quasi-steady (Brown & Cheng 1981).

Since the scale of the inner inviscid region on a blunt body is ϵ^3 (see (3.9), (4.29) and (4.30)) when the non-Brillouin-point term $\bar{a}\,\bar{x}^{\frac{3}{2}}$ is completely negligible there, it follows from (3.2), (3.6) and (7.1) that this occurs when

$$S \ll Re^{\frac{1}{64}}, \tag{7.6}$$

which, in view of the restriction $S \gg 1$, is certainly difficult to achieve in practice.

Cheng & Smith (1982) show that $\bar{a} = O\,(\epsilon_0^{\frac{1}{2}})$ independently of ϵ for laminar boundary layers even when the ratio ϵ of transverse to streamwise body dimensions is small. Thus $\bar{a} = O(\epsilon_0^{\frac{1}{2}})$, while $\bar{b}, \bar{c} = O(\epsilon)$ in the slender-body case. Since the distance between separation and Brillouin points is roughly determined by the relative magnitudes of the first and third terms in (3.6), it is now clear that this distance can (as indicated in § 1) remain $O(1)$ as $Re \to \infty$ if $\epsilon/\epsilon_0^{\frac{1}{2}}$ is held fixed in this limit, i.e. if $\epsilon = O(\epsilon_0^{\frac{1}{2}})$.

† It may then be necessary to include additional terms (involving $\bar{c}$) in the forced solution (5.13) and (5.14), but these will not contribute to the asymptotic form (5.18) that applies in the inner inviscid region, and therefore cannot effect the coupling coefficient, which is our only concern in this work.

8. Validity of the Kutta condition

We now show that the Kutta condition is satisfied to lowest order of approximation for quasi-steady laminar separations. We consider only the blunt-body case where the steady coefficient $\bar{a}$ is given by (7.1). Condition (7.3) insures that the unsteady interaction region for the complete (steady plus unsteady) solution in the moving reference frame will have the same triple-deck structure as was found for the steady case by Sychev (1972) and Smith (1977) to lowest order in ϵ_0. Though, as shown by Brown & Cheng (1981), unsteady foredecks must be incorporated in order to match it to the unsteady upstream boundary layer. In fact, the only change is that the scaled lower-deck velocity U (see Smith 1977) must now satisfy the wall boundary condition

$$U = \frac{-\alpha_0}{\epsilon_0\,\sigma^{\frac{1}{4}}}\operatorname{Re} \mathrm{i}x_s\,\mathrm{e}^{-\mathrm{i}t}, \tag{8.1}$$

which results from the motion of the separation point and leaves the triple-deck structure unchanged as long as $|x_s|\,\alpha_0 = O(\epsilon_0)$, which we now assume to be the case. (Recall that the coordinate system is attached to the separation point and that we can make the amplitude α_0 of the forcing be as small as we like.) Here σ is the scaled skin friction of the attached laminar boundary layer just ahead of the separation point.

Since (7.3) ensures that the time only enters as a parameter within the triple deck, it is easy to see from (2.4)–(2.6) of Smith (1977), say, that $a + (\alpha_0/\lambda\delta)\operatorname{Re} a_1\,\mathrm{e}^{-\mathrm{i}t}$ must be representable as a function of the form

$$\epsilon_0^{\frac{1}{2}}\,\sigma^{\frac{9}{8}} f\left(\frac{\alpha_0}{\epsilon_0\,\sigma^{\frac{1}{4}}}\operatorname{Re} \mathrm{i}x_s\,\mathrm{e}^{-\mathrm{i}t}\right). \tag{8.2}$$

Then assuming that $|x_s|\alpha_0 \ll \epsilon_0$ and that $f'(0) \neq 0, \infty$, and expanding for small $|x_s|\,\alpha_0/\epsilon_0$, we obtain

$$a = \bar{a} = \epsilon_0^{\frac{1}{2}}\,\sigma^{\frac{9}{8}} f(0) \tag{8.3}$$

and

$$a_1 = -\lambda\delta\,\frac{\sigma^{\frac{7}{8}}}{\epsilon_0^{\frac{1}{2}}} f'(0)\,\mathrm{i}x_s, \tag{8.4}$$

where the prime denotes a derivative with respect to the indicated argument.

Smith (1977) showed that

$$f(0) \approx \tfrac{2}{3} \times 0.44, \tag{8.5}$$

and, using (3.10) and (6.13) to eliminate a_1, we obtain

$$x_s = \frac{-\mu P_0'(0,0)}{\frac{1}{2}\sigma^{\frac{7}{8}}(\epsilon_0^{\frac{1}{8}}\,\lambda) f'(0)\,\mathrm{i} + \mu}, \tag{8.6}$$

where (4.28) and (8.3) show that $\epsilon_0^{\frac{1}{8}}\,\lambda = O(1)$ as $\epsilon_0 \to 0$ and

$$\mu \equiv (\epsilon_0^2/\epsilon)^{\frac{1}{3}} \tag{8.7}$$

is $\ll 1$ when the quasi-steady condition (7.3) holds.

Thus, as anticipated in §6, x_s is small relative to $P_0'(0,0)$, and the Kutta condition is therefore satisfied in this case. Although this result was, for simplicity, established only for the blunt-body case, it also applies to slender bodies because $\bar{a}$ is still independent of ϵ in this case.

9. Summary and discussion

The analysis was, in the main, restricted to the case where the separation point is not a Brillouin point. We have, however, shown (in §7) that this is the only case likely to be encountered in practice. The most important results of this work are the formulas for the initial amplitude C_0 of the instability wave or 'coupling coefficient': (6.15) in the general case and (6.2) when an unsteady Kutta condition is satisfied at the separation point. Even though these results were, for definiteness, derived for a particular source, they are actually independent of the detailed nature of that source and depend only on the amplitude $P_0'(0,t)\,\mathrm{e}^{\mathrm{i}t}$ of the streamwise pressure gradient that would be produced at the separation point in the absence of flow separation. That this is also true for instability waves generated at sharp trailing edges, was deduced from physical considerations by Morkovin & Paranjape (1971).

When the unsteady Kutta condition is satisfied, (6.2) and (4.28) imply that the coupling coefficient will be proportional to $\bar{a}^{\frac{4}{3}}$, where $\bar{a}$ is the lowest-order expansion coefficient in the equation (3.6) for the steady separated streamline near the separation point. We expect that the latter quantity will exhibit a rather complicated dependence on the global properties of the mean flow when the upstream boundary layer is turbulent, and we know that it is determined by a local interaction when the separation is laminar. In fact, it is then given by (8.4) and (8.6), where the only global dependence is through the scaled skin friction σ of the laminar boundary layer just upstream of the separation point. These results also show that the coupling coefficient is proportional to the Reynolds number to the $-\frac{1}{12}$ power in this case.

Equations (3.10), (3.11), (4.26) and (4.28) show that it is proportional to $\epsilon^{\frac{2}{3}} = 1/S^{\frac{2}{3}}$ for both slender and blunt bodies. Here $S = 1/\epsilon$ is given by (3.1). Thus the coupling coefficient should decrease with the frequency to the $-\frac{2}{3}$ power when the Kutta condition is satisfied, which should be the simplest of our results to actually verify experimentally.

The Kutta condition will be satisfied when the dimensionless amplitude x_s of the unsteady separation point motion is small relative to the unsteady pressure-gradient amplitude P_0', in which case (6.5) will reduce to (6.2). The relation between x_s and P_0' must be determined by analysing the strong viscous (and non-parallel flow) effects that result from the steep velocity gradients in the vicinity of the separation point. We did this for the important special case of quasi-steady laminar separation, which occurs when

$$\epsilon_0^2 \gg \epsilon.$$

We found that x_s and P_0' are then related by (8.6) and (8.7), which show that the Kutta condition is satisfied in this case. It is less likely to be satisfied at higher frequencies, since (8.7) shows that the dimensionless separation-point displacement amplitude x_s increases with frequency. Notice, however, that the dimensional separation-point displacement amplitude $l\epsilon x_s$ still exhibits the expected decrease in magnitude.

The Kutta condition is satisfied for unsteady flow at a sharp trailing edge even in the fully unsteady limit where $\epsilon = O(\epsilon_0^2)$ (Daniels 1977; Brown & Daniels 1975), but the results of §8 strongly suggest that this will not be true in the present case of separation from a smooth surface.

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